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Studies on Holonomic Quantum Fields. I

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To understanding the mathematical structure of quantized fields or systems with infinite freedom, non trivial but exactly calculable models would be of great help [1]. In this and subsequent notes we present, both in the continuum and in the lattice, 2-dimensional soluble models of neutral scalar massive field theory whose τ-functions exhibit a non trivial singularity structure.

In the present article we deal with the continuum case. We introduce an auxiliary free fermi/bose field and construct the field operator by giving its induced rotation in the space of wave functions. Making use of the "theory of rotation" (2. cf.[2]) developed recently by the first author, we express this field operator in the normal product form of these free fields. We also calculate the asymptotic fields and the S-matrix of the field φ^F defined in 3. Next we give explicit formulae for τ-functions of these models and study their holonomy structure.

The lattice field theory will be discussed in a subsequent paper. Specifically we shall show that our model φ^F/φ_F coincide with the scaling limit of the Ising model from above/below the critical temperature. Main part of these results has been announced in [3].

We use the following notations. The space-time and the energy-momentum co-ordinates are denoted by $x=(x^0,x^1)$ and

$p=(p^0,p^1)$. We also use $x^{\pm} = (x^0 \pm x^1)/2$ and $p^{\pm}=p^0 \pm p^1$. The mass-shell $\{p \in \mathbb{R}^2 \mid p^2=(p^0)^2-(p^1)^2=m^2\}$ $(m>0)$ is denoted by M. For $p \in M$ we set $u^{\pm 1}=p^{\pm}/m$, $\underline{du}=du/2\pi|u|$.

1. Let $\psi(u)^{\dagger}$ and $\psi(u)$ $(u>0)$ be the creation and annihilation operators of auxiliary fermion. If we define $\psi(u)=\psi(-u')^{\dagger}$ for $u<0$, their anti-commutation relation reads $[\psi(u),\psi(u)]_{+}=2\pi|u|\delta(u+u')$. Likewise we define auxiliary bosons $\phi(u)$ with the commutation relation $[\phi(u),\phi(u')]_{-}=2\pi u\delta(u+u')$. In two dimensional space-time these two are in fact equivalent. Namely

$$(1) \qquad \phi_{\pm}(u)=:\psi(u)\exp\int_0^{\infty}\theta(\pm(|u|-u'))\psi(u')^{\dagger}\psi(u')\underline{du}':$$

satisfy the commutation relation $[\phi_{\pm}(u),\ \phi_{\pm}(u')]_{-}=2\pi u\delta(u+u')$, and conversely $\psi(u)$ is given by

$$(2) \qquad \psi(u)=:\phi_{\pm}(u)\exp\int_0^{\infty}\theta(\pm(|u|-u'))\phi_{\pm}(u')^{\dagger}\phi_{\pm}(u')\underline{du}': \ .$$

2. We let W denote an orthogonal/symplectic space, a vector space equipped with a non-degenerate symmetric/skew-symmetric inner product $\langle w, w'\rangle$. First consider the orthogonal case and denote by $A(W)$ the enveloping algebra (Clifford algebra) over W with defining relation $[w,w']_{+} = \langle w,w'\rangle$. $G(W)$ denotes the Clifford group $\{g \in A(W) \mid \exists g^{-1},\ gWg^{-1}=W\}$. Let $g \longmapsto g^*$ denote the anti-automorphism of $A(W)$ characterized by $w^*=w$ for $w \in W$. Set $n(g)=g^*g=gg^*$ for $g \in G(W)$, and $g \longmapsto n(g)$ defines a group homomorphism $G(W) \longrightarrow GL(1)$. Let $W=V^{\dagger}\oplus V$ be a decomposition into two holonomic subspaces. This means that there exist a basis $\psi^{\dagger}=(\psi^{\dagger}_{\mu})$ of $V^{\dagger}$ and a basis $\psi=(\psi_{\mu})$ of V such that $\langle\psi^{\dagger}_{\mu},\psi^{\dagger}_{\nu}\rangle=0$, $\langle\psi_{\mu},\psi_{\nu}\rangle$

$=0$ and $\langle\psi^\dagger_\mu,\psi_\nu\rangle=\delta_{\mu\nu}$. Then $A(W)$ is a semi-direct product of two exterior algebras $\Lambda(V^\dagger)$ and $\Lambda(V)$, and a $\Lambda(V^\dagger)$-$\Lambda(V)$ -isomorphism N: $A(W)=\Lambda(V^\dagger)\cdot\Lambda(V)\longrightarrow\Lambda(W)=\Lambda(V^\dagger)\wedge\Lambda(V)$ such that $N(1)=1$ is determined uniquely. The image $N(g)\in\Lambda(W)$ we call the norm of g. (In physicist's notation $g=:N(g):$.) For $g\in G(W)$ $T_g:w\in W\longmapsto gwg^{-1}\in W$ is a rotation, an isomorphism which preserves the inner product. Let $T_g(\psi^\dagger,\psi)=(\psi^\dagger,\psi)\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}$. First assume that T_4 is invertible. Then we have the following expression of the norm of g.

$$(3)\qquad N(g) = \langle g\rangle\exp((1/2)\psi^\dagger T_2T_4^{-1}{}^t\psi^\dagger+\psi^\dagger({}^tT_4^{-1}-1)^t\psi+(1/2)\psi T_4^{-1}T_3{}^t\psi),$$

where $n(g)=\langle g\rangle^2\,(\det T_4)^{-1}$, and we regard $\psi^\dagger_\mu,\psi_\mu$ as elements of $\Lambda(W)$. Next we assume that $\dim \operatorname{Ker}T_4=1$, and choose $\psi_0^\dagger\in V^\dagger$, $\psi_0\in V$ and $w\in G(W)\cap W$ such that $T_g\psi_0=\psi_0^\dagger$, $w^2=1$ and $\langle w,\psi_0^\dagger\rangle=1$. Then $(T_{wg})_4$ is invertible and

$$(4)\qquad N(g) = \psi_0^\dagger N(wg)+N(wg)\psi_0\ .$$

Here we regard $\psi_0^\dagger$ and ψ_0 as elements of $\Lambda(W)$. Next consider the symplectic case, and define $A(W)$, $G(W)$, etc. with due modifications. In particular $w^*=iw$ for $w\in W$, and the norm of $g\in A(W)$ is defined as an element of the symmetric tensor algebra $S(W)$. Assuming that T_4 is invertible, we have

$$(5)\qquad N(g)=\langle g\rangle\exp((1/2)\phi^\dagger(-T_2T_4^{-1})^t\phi^\dagger+\phi^\dagger({}^tT_4^{-1}-1)^t\phi+(1/2)\phi T_4^{-1}T_3{}^t\phi),$$

with $n(g)=\langle g\rangle^2\det T_4$.

3. Let now W be the space of wave functions $w(x)=(w_+(x),w_-(x))$ satisfying the Dirac equation $\partial w_\pm(x)/\partial x^\pm \mp mw_\mp(x)=0$. An orthogonal structure is introduced to W by

defining $\langle w,w'\rangle=\int_{-\infty}^{+\infty}dx^1(w_+(x)w'_+(x)+w_-(x)w'_-(x))$. If we identify $w\in W$ with the operator $\int_{-\infty}^{+\infty}dx^1(w_+(x)\psi_+(x)+w_-(x)\psi_-(x))$, where $\psi_\pm(x)=(1/\sqrt{2})\int_{-\infty}^{+\infty}\underline{du}\sqrt{0+iu}^{\pm 1}\ \psi(u)\exp(-im(x^-u+x^+u^{-1}))$, the Clifford algebra $A(W)$ is nothing but the operator algebra of free fermions. We choose as $V^\dagger/V$ the set of creation/annihilation operators in W. Set $W_x^\pm=\{w\in W|\,w(x')=0$ if $(x'-x)^2<0$, $x'^1-x^1\lessgtr 0\}$, and we shall have $W=W_x^+\oplus W_x^-$, an orthogonal decomposisition. We now introduce our field operator $\varphi_F(x)\in A(W)$ by specifying its induced rotation $T_{\varphi_F(x)}$ with the property $T^2_{\varphi_F(x)}=1$ by

(6) $$T_{\varphi_F(x)}(w^++w^-)=w^+-w^-,\ w^\pm\in W_x^\pm.$$

Applying the formula (3) to the present situation and choosing $\langle\varphi_F(x)\rangle=1$ we obtain the following expression for $\varphi_F(x)$:

(7) $$\varphi_F(x)=:\exp L_F(x):,$$

where $L_F(x)=(1/2)\iint_{-\infty}^{+\infty}\underline{du}\ \underline{du'}\frac{-i(u-u')}{u+u'-i0}\psi(u)\psi(u')\exp(-im(x^-(u+u')+x^+(u^{-1}+u'^{-1})))$. The micro-causality and the Lorentz covariance of $\varphi_F(x)$ are manifest in this approach.

We construct $\varphi^F(x)$ and $\varphi_B(x)$ analogously, using the formulae in the case dim $\mathrm{Ker}T_4=1$ and in the symplectic case, respectively. In the latter case we choose as W the solution space to the Klein-Gordon equation and equip it with the inner product $\langle w,w'\rangle=-i\int_{-\infty}^{+\infty}dx^1(w(x)\cdot\partial w'(x)/\partial x^0-w(x)/\partial x^0\cdot w'(x))$. The results are

(8) $$\varphi^F(x)=:\psi_0(x)\exp L_F(x):,$$

where $\psi_0(x)=\int_{-\infty}^{+\infty}\underline{du}\psi(u)\exp(-im(x^-u+x^+u^{-1}))$,

(9) $$\varphi_B(x)=:\exp L_B(x):,$$

where $L_B(x)=(1/2)\iint_{-\infty}^{+\infty}\underline{du}\ \underline{du'}\ \frac{-2\sqrt{u-i0}\sqrt{u'-i0}}{u+u'-i0}\phi(u)\phi(u')$
$\exp(-im(x^-(u+u')+x^+(u^{-1}+u'^{-1})))$.

4. The asymptotic fields for φ^F are defined by

$$(10)\qquad \varphi_\pm(x) = \int_{-\infty}^{+\infty}\underline{du}\,\varphi_\pm(u)\exp(-ipx) \ ,$$

where $\varphi_\pm(u)=\lim_{t\to\pm\infty} i\int_{x^0=t} dx^1(\varphi^F(x)(\partial/\partial x^0)\exp(ipx)-(\partial/\partial x^0)$
$\varphi^F(x)\exp(ipx))$. We find that this limit coincides with $\phi_\pm(u)$ defined in 1. The asymptotic states $|>_\pm$ are related to the auxiliary fermion states $|>_F$ through the formulae

$$(11)\qquad |u_n,\cdots,u_1>_\pm = \prod_{i<j}\varepsilon(\pm(u_i-u_j))\,|u_n,\cdots,u_1>_F,$$

where $\varepsilon(u)$ stands for the signature of u. Accordingly the particle number is conserved, and the S-matrix in the n-particle state is given by $(-)^{n(n-1)/2}$ times the identity operator, showing that the maximum phase shift is attained in this model.

5. The n-point τ-function of an operator $\varphi(x)$ is expressed as follows:

$$\tau_n(p_1,\cdots,p_n)=\sum_{\text{permutations}}^{n!} T_{n-1}(p_1,p_1+p_2,\cdots,p_1+\cdots+p_{n-1})$$

$$\times\,(2\pi)^2\delta^2(p_1+\cdots+p_n)\ ,$$

(12) where

$$T_{n-1}(q_1,\cdots,q_{n-1}) = \sum_\nu (1/\nu_1!\cdots\nu_{n-1}!)$$

$$\int_0^\infty\!\!\cdots\!\int\underline{du}\ \prod_{j=1}^{n}\varphi_{\nu_j+\nu_{j-1}}(u_{j\nu_j},\cdots,u_{j1},-u_{j-1\,1},\cdots,-u_{j-1\nu_{j-1}})$$

$$\times\prod_{j=1}^{n-1}2\pi\delta(q_j^+-mU_j)\,i(q_j^--mU_j'+i0)^{-1}\ ,$$

with $U_j=\sum_{k=1}^{\nu_j}u_{jk}$, $U_j'=\sum_{k=1}^{\nu_j}u_{jk}^{-1}$ and $\nu_0=\nu_n=0$. The (anti-)

symmetric functions φ_n are the matrix elements defined by $\varphi_n(u_1,\dots,u_n)=\langle -u_{m+1},\dots,-u_n|\varphi(0)|u_1,\dots,u_m\rangle$ for $u_1,\dots,u_m>0$ and $u_{m+1},\dots,u_n<0$. In our models they are obtained from (7), (8) and (9).

$$(13)\quad \varphi_{F,n}(u_1,\dots,u_n) = \mathrm{Pfaffian}\left(iP\frac{u_j-u_k}{u_j+u_k}\right)_{1\leqq j,k\leqq n}$$

$$= \begin{cases} i^{n/2} \prod_{1\leqq j<k\leqq n} P\dfrac{u_j-u_k}{u_j+u_k} & (n\ \text{even}) \\ 0 & (n\ \text{odd}), \end{cases}$$

$$(14)\quad \varphi^F{}_n(u_1,\dots,u_n) = -i\varphi_{F,n+1}(\infty,u_1,\dots,u_n)$$

$$= \begin{cases} 0 & (n\ \ \text{even}) \\ i^{(n-1)/2} \prod_{1\leqq j<k\leqq n} P\dfrac{u_j-u_k}{u_j+u_k} & (n\ \ \text{odd}) , \end{cases}$$

and

$$(15)\quad \varphi_{B,n}(u_1,\dots,u_n) = \mathrm{Hafnian}\left(-2P\frac{\sqrt{u_j-i0}\sqrt{u_k-i0}}{u_j+u_k}\right)_{1\leqq j,k\leqq n} .$$

Here $P(1/u+v)$ denotes the principal value of $1/u+v$, and for a symmetric matrix $(a_{jk})_{1\leqq j,k\leqq n}$ we set $\mathrm{Hafnian}(a_{jk})=0$ for odd n and $=\Sigma' a_{j_1j_2}a_{j_3j_4}\cdots a_{j_{n-1}j_n}$ for even n, where the sum is taken over $(n-1)!!$ pairings of indices $1,\dots,n$. In particular the (Euclidean) two point functions of φ_F and φ^F coincide with those obtained by [4] and [5].

The singularity/holonomy spectrum of $\tau_n(p)$ is confined to the union of positive-α/complex Landau singularities corresponding to graphs with no internal vertices [6], where the number of (internal and external) lines incident to each vertex is always even for φ^F and is always odd for φ_F, φ_B. On the leading singularity Λ^+_G, the order of τ_n for φ^F or φ_F is given by

(16) $$\operatorname{ord}_{\Lambda^+_G} \tau_n = n_e - N/2 - \sum_{i<j} N_{ij}(N_{ij}-1)/2,$$

where n_e denotes the number of vertices of G, N_{ij} the number of internal lines joining the vertices i and j, and $N = \sum_{i<j} N_{ij}$. Note that repulsive effect of multiple internal lines is incorporated in (16).

Finally we remark that the generalized unitarity relation for the τ-function of φ^F

(17) $$0 = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=0}^{n} (-)^k \sum_{\text{combinations}}^{\binom{n}{k}} \int\cdots\int \prod_{i=1}^{\ell} du_i \, \tau_k^{(\ell)}(p_1,\cdots,p_k;u_1,\cdots,u_\ell) \times \overline{\tau_{n-k}^{(\ell)}(-p_{k+1},\cdots,-p_n;u_1,\cdots,u_\ell)}$$

where we set $\tau_k^{(\ell)}(p_1,\cdots,p_k;u_1,\cdots,u_\ell) = \tau_{k+\ell}(p_1,\cdots,p_k;q_1,\cdots,q_\ell) \times \prod_{i=1}^{\ell}(q_i^2-m^2)\big|_{q_i^{\pm}\mapsto u_i^{\pm 1}}$ and bar denotes the complex conjugation, is directly and analytically verified by using our explicit formulae (12) and (14).

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